Laplace Inverse :-

Method (1):- (From table)

ex:- $LT^{-1}\left\{\frac{1}{s}\right\} = 1$

ex:- $LT^{-1}\left\{\frac{1}{s^2}\right\} = t$

ex:- $LT^{-1}\left\{\frac{1}{s^{10}}\right\} = \frac{t^9}{9!}$ $\quad LT\{t^9\} = \frac{9!}{s^{10}}$ لأن

ex:- $LT^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{3!}$

ex: $LT^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$

ex: $LT^{-1}\left\{\frac{1}{s+6}\right\} = e^{-6t}$

ex: $LT^{-1}\left\{\frac{1}{(s-3)^4}\right\} = \frac{t^3}{3!} e^{3t}$ (الإزاحة أولاً ثم الدالة)

الإزاحة

ex: $LT^{-1}\left\{\frac{1}{(s-7)^{10}}\right\} = \frac{t^9}{9!} e^{7t}$

الإزاحة

ex: $LT^{-1}\left\{\frac{1}{(s^2-2s+1)}\right\} = \frac{1}{(s-1)^2} = t\, e^{t}$

الإزاحة بعكس الإشارة

ex:— $LT^{-1}\left\{\dfrac{1}{S^2+4S+4}\right\} = LT^{-1}\left\{\dfrac{1}{(S+2)^2}\right\}$

$$F(t) = t\,e^{2t}$$

ex: $LT^{-1}\left\{\dfrac{S+1}{(S+1)^2+\omega^2}\right\} = \cos\omega t\; e^{-t}$

ex: $LT^{-1}\left\{\dfrac{3}{(S-2)^2+9}\right\} = \sin 3t\; e^{2t}$

ex: $LT^{-1}\left\{\dfrac{S+2}{(S+2)^2-81}\right\} = \cosh 9t\; e^{-2t}$

ex: $LT^{-1}\left\{\dfrac{S+1}{(S+2)^2+4}\right\}$ → يجب ان يكون الإزاحة فوق وتحت متشابهين

$$LT^{-1}\left\{\frac{S+1+1-1}{(S+2)^2+4}\right\} = LT^{-1}\left\{\frac{S+2-1}{(S+2)^2+4}\right\}$$

$$LT^{-1}\left\{\frac{S+2}{(S+2)^2+4} - \frac{1}{(S+2)^2+4}\right\}$$

$$LT^{-1}\left\{\frac{S+2}{(S+2)^2+4} - \frac{1}{2}\,\frac{2}{(S+2)^2+4}\right\}$$

$$f(t) = \cos 2t\, e^{-2t} - \frac{1}{2}\sin 2t\, e^{-2t}$$

ex: هام جدا

$$LT^{-1}\left\{\frac{1}{S^2+4S+9}\right\}$$

اكمال مربع

$$S^2+4S+4-4+9$$
$$S^2+4S+4+5$$
$$(S+2)^2+5$$

$$LT^{-1}\left\{\frac{1}{(S+2)^2+5}\right\}$$

$$\frac{1}{\sqrt5}LT^{-1}\left\{\frac{\sqrt5}{(S+2)^2+5}\right\}=\frac{1}{\sqrt5}\sin\sqrt5 t\, e^{-2t}$$

ex: هام جدا

$$LT^{-1}\left\{\frac{S+1}{S^2+6S-4}\right\}$$

$$S^2+6S-4$$
$$S^2+6S+9-9-4$$
$$(S+3)^2-13$$

$$LT^{-1}\left\{\frac{S+1}{(S+3)^2-13}\right\}$$

$$LT^{-1}\left\{\frac{S+1+2-2}{(S+3)^2-13}\right\}=LT^{-1}\left\{\frac{S+3-2}{(S+3)^2-13}\right\}$$

$$=LT^{-1}\left\{\frac{S+3}{(S+3)^2-13}-\frac{2}{(S+3)^2-13}\right\}$$

$$=LT^{-1}\left\{\frac{S+3}{(S+3)^2-13}-\frac{2}{\sqrt{13}}\,\frac{\sqrt{13}}{(S+3)^2-13}\right\}$$

$$f(t)=\cosh\sqrt{13}t\, e^{-3t}-\frac{2}{\sqrt{13}}\sinh\sqrt{13}t\, e^{-3t}$$

$$\boxed{\begin{aligned} &\because LT\{t F(t)\} = -\frac{d}{ds} f(s) \\ &\therefore LT^{-1}\left\{\frac{d}{ds} f(s)\right\} = -tF(t) \end{aligned}}$$

هامة

مسألة هامة

ex: $LT^{-1}\{\cot^{-1} s\}$

$f(s) = \cot^{-1} s$

$$\frac{df(s)}{ds} = -\frac{1}{s^2+1}$$

$$LT^{-1}\left\{\frac{df(s)}{ds}\right\} = -LT^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$-tF(t) = -\sin t$

$$\therefore \boxed{F(t) = \frac{\sin t}{t}}$$

ex: $LT^{-1}\{\tan^{-1} s\}$

$f(s) = \tan^{-1} s$

$$\frac{df(s)}{ds} = \frac{1}{s^2+1}$$

$$LT^{-1}\left\{\frac{df(s)}{ds}\right\} = LT^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$-t\,F(t) = \sin t$

$$\boxed{F(t) = -\frac{\sin t}{t}}$$

ex: $LT^{-1}\{\tan^{-1}(s+1)\}$

$F(s) = \tan^{-1}(s+1)$

$$\frac{df(s)}{ds} = \frac{1}{(s+1)^2+1} \gg LT^{-1}\left\{\frac{df(s)}{ds}\right\} = LT^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$-tF(t) = \sin t\, e^{-t}$

$$\boxed{F(t) = -\frac{\sin t\, e^{-t}}{t}}$$

ex:- $LT^{-1}\left\{\ln \frac{s^2+1}{s^2-1}\right\}$

$$f(s) = \ln\left[\frac{s^2+1}{s^2-1}\right] = \ln(s^2+1) - \ln(s^2-1)$$

$$\frac{df(s)}{ds} = \frac{2s}{s^2+1} - \frac{2s}{s^2-1}$$

$$LT^{-1}\left\{\frac{df(s)}{ds}\right\} = 2\,LT^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\,LT^{-1}\left\{\frac{s}{s^2-1}\right\}$$

$$-t\,F(t) = 2\cos t - 2\cosh t$$

$$\boxed{F(t) = -\frac{2\cos t - 2\cosh t}{t}}$$

ex:- $LT^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\}$

$$f(s) = \ln\left(1+\frac{1}{s}\right) = \ln\left(\frac{s+1}{s}\right)$$

$$f(s) = \ln(s+1) - \ln s$$

$$\frac{df(s)}{ds} = \frac{1}{s+1} - \frac{1}{s}$$

$$-t\,F(t) = \bar{e}^{t} - 1$$

$$\boxed{F(t) = \frac{\bar{e}^{t} - 1}{-t}}$$

Method(2): *partial Fraction* الكسور الجزئية

$$\frac{f(s)}{g(s)} = \frac{f(s)}{s(s-1)(s+3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}$$

جذور (roots)

partial Fraction

كسور جزئية

1 Not repeated roots:- جذور غير مكررة :-

$$\frac{f(s)}{(s-1)(s+10)} = \frac{A}{(s-1)} + \frac{B}{(s+10)}$$

أصفار المقام غير مكررة وتسمى (roots)

طريقة الفرض.

2 Repeated roots:- جذور مكررة :-

$$\frac{f(s)}{(s-1)^3} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

3 Complex roots:- جذور تخيلية :-

$$\frac{f(s)}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

طريقة الفرض في حالة Complex

$as^2+bs+c=0 \Rightarrow$ if $b^2-4ac<0$

Complex roots

ex: Find $LT^{-1}\left\{\frac{1}{S(S^2+1)}\right\}$

$$\frac{1}{S(S^2+1)} = \frac{A}{S} + \frac{BS+C}{S^2+1}$$

$$\boxed{1 = A(S^2+1) + (BS+C)S}$$

$\boxed{S=0}$: $1 = A \gg \boxed{A=1}$

$\boxed{\text{Coeffs } S^2}$: $0 = A + B \gg \boxed{B=-1}$

$\boxed{\text{Coeffs } S^1}$: $0 = 0 + C \gg \boxed{C=0}$

$$\therefore\ LT^{-1}\left\{\frac{1}{S} - \frac{S}{S^2+1}\right\} = 1 - \cos t$$

ex: $LT^{-1}\left\{\frac{1}{(S-2)(S-1)(S+3)}\right\}$

$$\frac{1}{(S-2)(S-1)(S+3)} = \frac{A}{S-2} + \frac{B}{S-1} + \frac{C}{S+3}$$

$$\boxed{1 = A(S-1)(S+3) + B(S-2)(S+3) + C(S-2)(S-1)}$$

$\boxed{S=1}$: $1 = -4B \gg \boxed{B = -\frac{1}{4}}$

$\boxed{S=2}$: $1 = 5A \gg \boxed{A = \frac{1}{5}}$

$\boxed{S=-3}$: $1 = 20C \gg \boxed{C = \frac{1}{20}}$

$$LT^{-1}\left\{\left(\frac{\frac{1}{5}}{S-2} - \frac{\frac{1}{4}}{S-1} + \frac{\frac{1}{20}}{S+3}\right)\right\} = \frac{1}{5}e^{2t} - \frac{1}{4}e^{t} + \frac{1}{20}e^{-3t}$$

ex: $LT^{-1}\left\{ \dfrac{s+1}{(s-1)^2(s^2+1)} \right\}$

$$\frac{s+1}{(s-1)^2(s^2+1)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+1}$$

$$s+1 = A(s-1)(s^2+1) + B(s^2+1) + (Cs+D)\underbrace{(s-1)^2}_{s^2-2s+1}$$

$\boxed{s=1}$: $2 = 2B \Rightarrow \boxed{B=1}$

$\boxed{\text{coeff } s^3}$: $0 = A + C \Rightarrow \boxed{A = -C}$

$\boxed{\text{coeff } s^2}$: $0 = -A + B - 2C + D$

$\boxed{\text{coeff } s^1}$: $1 = A + C - 2D \Rightarrow \boxed{D = -\frac{1}{2}}$

$0 = C + 1 - 2C - \frac{1}{2} \Rightarrow \boxed{C = \frac{1}{2}}$

$\boxed{A = -\frac{1}{2}}$

$$LT^{-1}\left\{ \frac{-\frac{1}{2}}{(s-1)} + \frac{1}{(s-1)^2} + \left(\tfrac{1}{2}\right)\frac{s}{s^2+1} - \left(\tfrac{1}{2}\right)\frac{1}{s^2+1} \right\}$$

$$F(t) = -\frac{1}{2}e^t + te^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

Applications:
The Solution of the ordinary DE :-

Theorem:

تستخدم هذه النظرية لحل المعادلات التفاضلية

$$LT\{F(t)\} = f(s)$$
$$LT\{F'(t)\} = Sf(S) - F(0)$$
$$LT\{F''(t)\} = S^2 f(s) - SF(0) - F'(0)$$
$$LT\{F'''(t)\} = S^3 f(s) - S^2 F(0) - S F'(0) - F''(0)$$

$$LT\{y(t)\} = Y(s)$$
$$LT\{y'(t)\} = S\,Y(s) - y(0)$$
$$LT\{y''(t)\} = S^2 Y(s) - Sy(0) - y'(0)$$

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ex: solve $y'' + 3y' + 2y = 3e^{-2t}$, $y(0) = y'(0) = 0$

$$LT\{y''\} + 3LT\{y'\} + 2LT\{y\} = 3\,LT\{e^{-2t}\}$$

$$S^2 Y(s) - S\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,0} + 3[SY(s) - \cancel{y(0)}^{\,0}] + 2Y(s) = \frac{3}{S+2}$$

$$Y(s)[S^2 + 3S + 2] = \frac{3}{S+2} \gg Y(s) = \frac{3}{(S+2)(S^2+3S+2)}$$

$$\therefore Y(s) = \frac{3}{(S+2)^2(S+1)} = \frac{A}{S+2} + \frac{B}{(S+2)^2} + \frac{C}{(S+1)}$$

$$3 = A(S+2)(S+1) + B(S+1) + C(S+2)^2$$

$S=-2$: $3 = -B \gg \boxed{B=-3}$

$S=-1$: $\boxed{3 = C}$

coeff S^2: $0 = A + C \gg \boxed{A=-3}$

$$\therefore Y(s) = \frac{-3}{S+2} - \frac{3}{(S+2)^2} + \frac{3}{S+1}$$

$$\boxed{y(t) = -3e^{-2t} - 3te^{-2t} + 3e^{-t}}$$

ex:- solve the following differential equations using LT

(1) $y'' + 4y' + 3y = 0 \;,\; y(0) = 3 ,\; y'(0) = 1$

sol:

$LT\{y''\} + 4\,LT\{y'\} + 3\,LT\{y\} = 0$

$s^2 Y(s) - s\,y(0) - y'(0) + 4s\,Y(s) - 4\,y(0) + 3\,Y(s) = 0$

$Y(s)\left[s^2 + 4s + 3\right] = 3s + 1 + 12 = 3s + 13$

$$Y(s) = \frac{3s+13}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$\boxed{3s + 13 = A(s+3) + B(s+1)}$$

$\boxed{s=-1}$: $10 = 2A \rightarrow \boxed{A = 5}$

$\boxed{s=-3}$: $4 = -2B \rightarrow \boxed{B = -2}$

$$Y(s) = \frac{5}{s+1} - \frac{2}{s+3}$$

$\therefore\; y(t) = 5e^{-t} - 2e^{-3t}$

(2) $y'' + y = 1 \;,\; y(0) = 6 ,\; y'(0) = 0$

sol: $LT\{y''\} + LT\{y\} = LT\{1\}$

$s^2 Y(s) - s\,y(0) - \cancel{y'(0)} + Y(s) = \frac{1}{s}$

$Y(s)\left[s^2 + 1\right] = \frac{1}{s} + 6s = \frac{6s^2+1}{s}$

$$Y(s) = \frac{6s^2+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+c}{s^2+1}$$

$$6s^2+1 = A(s^2+1) + (Bs+c)\,s$$

$\boxed{s=0}$: $\boxed{1=A}$

$\boxed{\text{coeff } s^2}$: $6 = A + B \gg \boxed{B=5}$

$\boxed{\text{coeff } s^1}$: $\boxed{0=C}$

$$Y(s) = \frac{1}{s} + 5\,\frac{s}{s^2+1} = 1 + 5\cos t$$

<u>ex1-(3)</u> $y'' - 5y' + 6y = e^{-t}$, $y(0)=0$, $y'(0)=2$

<u>sol:</u> $LT\{y''\} - 5\,LT\{y'\} + 6\,LT\{y\} = LT\{e^{-t}\}$

$$s^2Y(s) - s\underbrace{y(0)}_{0} - y'(0) - 5\,sY(s) + 5\underbrace{y(0)}_{0} + 6Y(s) = \frac{1}{s+1}$$

$$Y(s)[s^2 - 5s + 6] = \frac{1}{s+1}$$

$$Y(s) = \frac{1}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$1 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$$

$\boxed{s=-1}$: $1 = 12A \gg \boxed{A = \frac{1}{12}}$

$\boxed{s=2}$: $1 = -3B \gg \boxed{B = -\frac{1}{3}}$

$\boxed{s=3}$: $1 = 4C \gg \boxed{C = \frac{1}{4}}$

$$Y(s) = \frac{\frac{1}{12}}{s+1} - \frac{\frac{1}{3}}{s-2} + \frac{\frac{1}{4}}{s-3}$$

$$y(t) = \frac{1}{12}e^{-t} - \frac{1}{3}e^{2t} + \frac{1}{4}e^{3t}$$

Integration

Q3	Idea (30)%	Steps (40)%	Calculations (20)%	Final Result (10)%	Mark (25)

Evaluate the following integrals

(a) (i) $\int \frac{1}{\cos x} dx,$ (ii) $\int \frac{5}{\sqrt{9-4x^2}} \quad dx,$

(b) (i) $\int \frac{1}{\sin^4 x} dx,$ (ii) $\int \frac{1}{\sin x - 2\cos x} dx$

(c) (i) $\int e^{ax} \sin bx \, dx,$ (ii) $\int \frac{\sin x \cos^3 x}{1+\cos^2 x} dx$

(d) Evaluate $I_n = \int (\sin^{-1} x)^n \quad dx,$ and hence Find the value of I_4.

Q3	Idea (30)%	Steps (40)%	Calculations (20)%	Final Result (10)%	Mark (25)

Evaluate the following integrals

(a) (i) $\int \frac{x-1}{\sqrt{x^2+2}} dx,$ (ii) $\int \sinh^2 x \cosh^3 x \, dx$

(b) (i) $\int \frac{x}{\sqrt{2x^2-4x-7}} dx,$ (ii) $\int \frac{\sqrt{x}+1}{\sqrt{x}(\sqrt[3]{x}+1)} dx$

(c) (i) $\int \frac{x^4-3}{x^2+2x+1} dx,$ (ii) $\int_0^{\infty} \frac{1}{(1+x^2)^2} dx$

(d) Determine the area of the region enclosed by $y = \ln x$ and $x = 4$ and the x-axis

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Good luck!