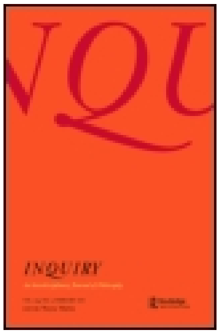

Inquiry: An Interdisciplinary Journal of Philosophy
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/sinq20

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Published online: 29 Aug 2008.

To cite this article: Michael Dummett (1982) Frege and Kant on geometry, Inquiry: An Interdisciplinary Journal of Philosophy, 25:2, 233-254

To link to this article: http://dx.doi.org/10.1080/00201748208601964

Inquiry, 25, 233–54

Discussion:

Frege and Kant on Geometry

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In his *Grundlagen*, Frege held that geometrical truths are synthetic *a priori*, and that they rest on intuition. From this it has been concluded that he thought, like Kant, that space and time are *a priori* intuitions and that physical objects are mere appearances. It is plausible that Frege always believed geometrical truths to be synthetic *a priori*; the virtual disappearance of the word 'intuition' from his writings from after 1885 until 1924 suggests, on the other hand, that he became dissatisfied with the notion of intuition as he had employed it in *Grundlagen*. The belief that *a priori* intuition is a source of knowledge does not in itself entail idealism: that is a question about what it is that makes true the propositions known in this way. In *Grundlagen*, Frege expressly states that geometrical truths are objective in the sense of being independent of our intuition. This shows that, even at that period, Frege did not draw the idealist conclusion drawn by Kant.

1

Hans Sluga has maintained that, on the realist/idealist axis, 'Frege's position must be akin to' that of Kant,[1] i.e. that he was a kind of transcendental idealist. His principal ground is that 'throughout his life Frege held to the Kantian thesis that space and time are *a priori* intuitions and that geometrical and temporal propositions are, therefore, synthetic *a priori*'; and he comments that anyone who commits himself to the Kantian thesis in question 'commits himself to something like the belief that objects, in the normal, empirical sense, are mere appearances and do not exist apart from cognition'.[2] Likewise he declares, in his book on Frege, that 'he held a Kantian view of space and hence a transcendentally subjective view of the objects that occupy it'.[3]

In one respect, it is very easy to evaluate such claims. References to philosophers, rather than to mathematicians and logicians, are extremely sparse in Frege's other writings; but they are plentiful in *Grundlagen*. He is there at particular pains to clarify his philosophical position *vis-à-vis* that of Kant. It is surely for this reason that, in that work, he makes extensive use of the Kantian terms 'analytic', 'synthetic', '*a priori*' and '*a posteriori*', which virtually never occur in his other writings.[4] While he was concerned to determine the grounds of our knowledge, and particularly of our knowledge of arithmetic (number theory and analysis), he was not deeply interested in the notion of necessity. His comparison, in *Grundlagen*, of his own views with those of Kant is not merely tacit, however: he endeavoured, in many passages, to make it explicit.

In another respect, we face a certain difficulty in evaluating Sluga's remarks. His reference to 'temporal propositions' is gratuitous, since Frege nowhere says anything about them;[5] but, although there are scattered observations about geometry in several of his writings, usually by way of contrast with arithmetic, it is difficult to elicit from them any very definite doctrine. The two series of articles, of 1903 and 1906, on the foundations of geometry, are of little help, preoccupied as they are with general questions about the nature of axioms and the sense, if any, in which an axiom-system implicitly defines the primitive terms. Since, after *Grundlagen*, Frege never used the terms 'analytic' and 'synthetic *a priori*', it is somewhat tendentious to describe him as holding throughout his life to the thesis that geometrical propositions are synthetic *a priori*. In *Grundlagen* itself, the principal thesis of the book is stated to be that 'arithmetical laws are analytic judgments';[6] in *Grundgesetze*, on the other hand, it is cited as having been that 'arithmetic is a branch of logic',[7] a stronger thesis, of course also maintained in *Grundlagen*. In respect of geometry, it is the other way about. In *Grundlagen*, Frege expressed his agreement with Kant that the truths of geometry are synthetic *a priori*, and praised Kant for having propounded that thesis.[8] But, except in the very late writings composed in the last year of his life, Frege never later made so definite a claim. He did, indeed, declare, in one of his notes to Jourdain's article about him, that 'the truths of geometry, in particular the axioms, are not facts of experience, at least if by that is meant that they are founded on sense-perceptions';[9] so we can confidently say that he did not regard them as *a posteriori* in Kant's sense. It is, undoubtedly, very unlikely that he at any time considered them to be analytic in the sense of *Grundlagen*: but it is going very far beyond the demonstrable facts to assert that, throughout his life, he held them to be synthetic *a priori*; we have no positive reason to affirm, though no specific reason to doubt, that he continued to view the Kantian trichotomy analytic/synthetic *a priori/a posteriori* with favour.

It is, nevertheless, quite likely to be true that Frege always regarded the truths of geometry as synthetic *a priori*. That he always connected our knowledge of them with intuition in anything like a Kantian sense is very much more dubious. It is doubtful whether he at any time subscribed, as Sluga alleges, to 'the Kantian thesis that space and time are *a priori* intuitions'; but from 1873, the year of his first publication, to 1885, when he delivered the lecture on formal theories of arithmetic, he certainly held that our knowledge of geometrical truths rests on intuition. During that period, he operated with a notion of intuition very similar to Kant's and embodying a certain epistemological conception. After that year, however, occurrences of the word 'intuition' (*Anschauung*) in his writings become very rare indeed. It occurs, in a negative context, in the opening sentence of *Grundgesetze*, where he says that he had sought, in *Grundlagen*, 'to make it probable that arithmetic is a branch of logic and does not need to borrow any ground of proof either from experience or from intuition'.[10] In the unpublished 'Logik' of 1897, he draws a distinction between an 'idea' (*Vorstellung*) and an 'intuition' (*Anschauung*): 'by an idea I understand an imaginative picture, which does not consist of present sensations, like an intuition, but of reawakened traces of past sensations or activities.'[11] Here '*Anschauung*' is used in a far more restricted sense than in *Grundlagen*, and indeed the English translators render it 'perception';[12] the passage therefore supplies positive evidence that Frege no longer adhered to the notion of intuition that he had employed in that book. The word occurs again in a letter to Hilbert of 1900, in which Frege remarks that 'it seems to me that you want to detach geometry entirely from spatial intuition and to turn it into a purely logical science like

arithmetic';[13] here, for all the context reveals, it could mean the same as in the 1897 'Logik'. Save in reference to a use of it by Hilbert, it appears in the 1903 'Grundlagen der Geometrie' only in the evasive remark, 'The question on what the justification rests for taking axioms to be true will not here be gone into: as the source for geometrical axioms, intuition is most often cited'.[14] Even in the very late writings, there is only one explicit reference to intuition, where it is equated with the geometrical source of knowledge,[15] about which Frege speaks a great deal, but of whose nature he tells us little. These late writings should not, of course, be taken as a guide to Frege's opinions at any earlier period, representing as they do a wholly new approach on his part to the philosophy of mathematics; we should note, however, that, in the fragment 'Zahl' of 1924, he observes that 'even the objects of geometry, points, lines and planes, etc., are not strictly speaking perceptible by the senses'.[16] From all this it appears highly probable that Frege came, after 1885, to reject as misconceived that notion of intuition which he had inherited from Kant and had made use of in *Grundlagen* and earlier writings, perhaps returning to something resembling it in 1924. That is, indeed, no more than surmise: what is quite certain is that we have no warrant whatever to ascribe to him, in the years 1886–1923, any positive views connecting our knowledge of geometry with intuition. (See Note Added in Proof on p. 254).

It nevertheless remains of interest to inquire whether Sluga's interpretation can be sustained for that period, from 1873 to 1885, when Frege did employ a notion of intuition and did make positive, though passing, observations concerning geometry. The question may seem unimportant; after all, Frege said even less about physical objects than he did about space. It would be a mistake to dismiss it in this fashion. *Grundlagen* already enunciates Frege's crucial thesis that not only what is actual (*wirklich*) is objective, and that numbers, in particular, though not actual, are objective.[17] For him, physical objects are not only the paradigm example of actual things, but equally the paradigm of what is objective. Naturally, since he maintained that 'objective' was a broader term than 'actual', he cites other examples, such as the axis of the Earth, to illustrate the difference between the two features; but, whenever he is wishing to assert the objectivity of numbers or of thoughts, without stressing their lack of actuality, it is to the material world that he turns for comparison. The mathematician can only discover what is there, just as can the geographer;[18] a judgeable content is objective, just as the Sun is objective;[19] a law of nature held good before it was discovered, just as an island in the ocean was there before it was seen by any human being.[20] If, then, Frege's view of physical objects were, or ever had been, as Sluga maintains, a 'transcendentally subjective' one, that would necessarily colour our understanding of what he meant by affirming thoughts and logical objects such as numbers to be as objective as they.

2

Whether Sluga is right to characterize the thesis expressed by Frege in *Grundlagen* by saying that geometrical truths are synthetic *a priori* as 'Kantian' depends upon whether he used 'synthetic *a priori*' in the same sense as Kant. In the first footnote to §3 of *Grundlagen*, Frege professes not to be assigning new senses to the terms 'analytic', 'synthetic', '*a priori*' and '*a posteriori*', but only to be hitting off 'what earlier writers, Kant in particular, have intended' by them. In §88, on the other hand, he is quite willing to say that Kant defined 'analytic' too narrowly, and to

speak of 'the wider concept' that he himself has employed, and this seems to accord much better with the facts of the case. He further says that 'on the basis of [Kant's] definition, the division of judgments into analytic and synthetic is not exhaustive'. If so, it would be necessary, in order to make it exhaustive, to redefine at least one of the two terms, though possibly not both; at any rate, it cannot be assumed without argument that Frege meant the same by 'synthetic *a priori*' as did Kant.

Frege falsely believed that every proof must have initial premisses. He believed this because he recognized no rule of inference which discharges a hypothesis, that is, whose conclusion depends on fewer hypotheses than those on which the premisses of the inference depend; although we frequently employ such forms of inference in informal argument, it was not until Gentzen that they came to be recognized and systematically treated by logicians. Given Frege's mistaken belief, it becomes impossible to comply with St. Paul's admonition to 'prove all things'. The question then is whether it is possible to comply with the weaker admonition, 'Prove everything that can be proved'. One possibility would be that every proposition that we can know to be true is capable of proof. If that were so, even compliance with the weaker admonition would be impossible: in order to give any proofs at all, we should have to accept as true some propositions which, though capable of proof, we had not proved. Frege rejected this possibility, and so conceived of the weaker admonition as capable of being followed. He held that there are some propositions which are intrinsically incapable of proof, but need no proof for us to recognize them as true. These are of two kinds: 'general laws, which . . . neither need nor admit of proof'; and 'facts, i.e. unprovable truths that lack generality, whose contents are predications concerning particular objects'.[21] On these two kinds of knowable but unprovable truths all our knowledge rests.

Frege defines an *a priori* proposition as one which is capable of being proved, by appeal only to definitions of the terms involved, from fundamental general laws, i.e. ones neither needing nor admitting of proof: he obviously intends that such a proof shall be of a purely deductive or logical character. An *a posteriori* truth is one whose proof necessarily involves some appeal to 'facts' (in the special sense of this word explained above). Among *a priori* truths, the analytic ones are distinguished by its being possible to prove them from 'general logical laws' alone, together with the definitions to which one may always appeal; the synthetic ones are those which cannot be proved without appealing to 'truths which are not of a general logical nature, but relate to a particular domain of knowledge'. Thus the general laws which neither need nor admit of proof can be divided into those which are logical in character and those which are not, the logical ones being those whose range of application is unrestricted. The definitions display an uncharacteristic carelessness, in that no provision is made for the status of those truths which are incapable of proof: obviously the fundamental laws of logic should be included among the analytic truths, fundamental general laws which are not logical among synthetic *a priori* ones, and particular 'facts' among *a posteriori* truths. All this is set out in §3 of *Grundlagen*. In the second footnote to that section, Frege argues that we must admit the existence of fundamental laws if we are to acknowledge any general truths,[22] since nothing follows from an individual fact save on the ground of a general law. He instances, as an example of such a fundamental (non-logical) general law, 'the general proposition that this procedure [induction] can establish the truth or at least the probability of a law'. This is the only quite specific example of such a non-logical fundamental law ever cited by Frege.

As everybody knows, the fundamental problem of the *Kritik der reinen Vernunft* is how synthetic *a priori* judgments are possible. This problem is, in the first instance, an epistemological one: the problem is not, at the outset, what makes synthetic *a priori* propositions true, but how we are able to judge them to be true. Now Frege was a logician rather than an epistemologist. Some commentators, such as Gregory Currie, have called this characterization in question on the ground of his concern with the basis of our knowledge of arithmetical truths. That was indeed one of his principal concerns; but it was properly his concern as a logician, since he proposed to justify those truths by deriving them from the fundamental laws of logic.

> The grounds which justify the recognition of a truth often lie in other truths already recognized. But if truths are known by us at all, this cannot be the only kind of justification. There must be judgments whose justification rests on something else, if indeed they require a justification at all. It is in this that the task of epistemology lies. Logic has to do only with those grounds of judgment which are truths. To judge when one is aware of other truths as grounds of justification is called *inferring*. There are laws governing this kind of justification, and it is the aim of logic to set out these laws of correct inference.[23]

The justification of a judgment, whether analytic, synthetic *a priori* or *a posteriori*, that is capable of proof is the proper concern of a logician, since proof is his subject-matter. But, since any proof has initial premisses, this will never be more than a relative justification, relative, namely, to the justification of the initial premisses. At least, that will be so if those initial premisses, which are either fundamental general laws or particular 'facts', require any justification. If they are general logical laws, then, again, the question whether they need a justification, and, if so, what it can be, is obviously one for the logician. The question is one for the epistemologist if they are non-logical laws or particular 'facts'.

Of such 'facts', and of the fundamental general laws, Frege says in §3 of *Grundlagen* that they do not admit of proof. But he also says that they do not need proof. This leaves it in question whether it is possible to recognize that they need no proof without adverting to the principle that you cannot be in need of that which it is impossible in principle for you to have: could we recognize that these truths needed no proof in advance of realizing that none was possible? It is difficult to see how one could do so save by recognizing that they needed no justification whatever; but Frege's text leaves it open whether that is so, or whether they can be justified by one of those other means referred to in the passage quoted above from the 'Logik' of the 1880s, means which involve neither proof nor appeal to other truths. On the face of it, one would be disposed to say that the individual 'facts' could be justified by perception or observation: and this leaves it open that the general laws, whether logical or non-logical, can be justified by reference to other faculties of ours for the direct apprehension of truth.

As regards the fundamental laws of logic, Frege gives no answer to this question in *Grundlagen*; but he does do so in the Preface to *Grundgesetze*. 'The question why and with what right we acknowledge a law of logic to be true, logic can answer only by reducing it to another law of logic. Where that is not possible, the question must remain unanswered.'[24] This is quite explicit: no justification whatever can be given for accepting those laws of logic which cannot be derived from other laws. If someone denies one of them, one can only say, 'We have here a hitherto unknown type of madness'.[25] Frege goes on to reject a particular putative justification.

> Stepping aside from logic, one may say: we are compelled to make judgments by our own nature and by external circumstances, and, when we judge, we cannot reject this law – e.g. that of identity – but must acknowledge it, unless we wish to bring our thought into confusion and finally renounce all judgment. I do not wish either to dispute or to endorse this view; I wish only to observe that we do not here have a logical deduction. What is given is not a ground for the law's being true, but for our taking it to be true. Moreover, this impossibility of rejecting the law, which constrains us, does not in the least prevent us from supposing beings who do reject it; but it prevents us from supposing that such beings are right to do so; and it also prevents us from doubting whether it is we or they who are right.[26]

Frege has already insisted on the sharpest distinction between being true and being taken to be true: 'being true is something different from being taken to be true, whether by one person, by many or by all, and can in no way be reduced to it.'[27] The fact, if it be a fact, that we cannot but take the fundamental laws of logic to be true, is no ground for their truth, for their being what we take them to be: in the nature of the case, no ground for that can be given, and hence no justification for our taking them to be true, if by a justification be meant something that goes to show that we are right to do so. This should not, however, push us into logical relativism. It must not be concluded that these laws are only true for us: the predicate 'true' does not admit of such a qualification. It is incoherent to say that we cannot but take the laws to be true, and in the same breath to disparage them as only true for us on the ground that other beings might reject them; if we are compelled to take them as true, then we take them to be *true*, and must thereby regard anyone who denies them as wrong.

The question is, therefore, how the matter stands with those fundamental general laws which are not logical in character, and particularly those which relate to space and underlie the science of geometry. Kant believed that he could explain how we come by a knowledge of such synthetic *a priori* truths. If Frege believed the same, and accepted the same explanation, then he was, in the respect that interests Sluga, a Kantian; if he believed that it could not be explained, or that there was a quite different explanation, he was not. Neither in Kant's nor in Frege's definitions of the terms 'synthetic' and '*a priori*' is there any appeal to a positive thesis about how we come by synthetic *a priori* knowledge, save by deduction from more fundamental synthetic *a priori* truths. Hence, simply to believe that geometrical truths are synthetic *a priori* does not make one a transcendental idealist: all depends upon the explanation of our knowledge of those truths. Specifically, it depends on whether it is our knowledge of them, or at any rate some feature of our constitution, that makes them true, or whether, as on the realist view, they are true quite independently of us.

3

In the 'Logik' of the 1880s, Frege suggests, as we have seen, that there are non-deductive justifications for certain truths. It would have been consistent for him to hold that judgments based on sense-perception are justified simply by being so based. Admittedly, although it is by means of our logical faculties or our reason that we acknowledge the truth of the fundamental laws of logic,[28] Frege does not say that they are justified by being directly apprehended by the reason, but that they are incapable of justification at all. Reason is not, however, a true parallel

to sense-perception: to ascribe reason to an individual just is to ascribe to him a capacity to grasp thoughts and to make judgments on the basis of other judgments from which they follow. Reason does not *prompt* us to do such things, but *consists in* our doing them, whereas sense-perception prompts us to make observational judgments rather than itself consisting in our making them. Frege insisted on the distinction between what brings about and what justifies a judgment:[29] with what right, then, might what prompted a judgment be cited in justification of it? There can be only two possibilities. Either we assume, or have reason to believe, that what prompts the judgment is a reliable sign that it is true; or we take the judgment as relating solely to the occurrence of that which prompts it. The former is the realist option, the latter the idealist one. For the realist, sense-perception supplies a justification for observational judgments because it is intrinsic to the concept of perception that to perceive things is a ground for taking them to be as we perceive them, and because this presupposition is confirmed by our ability to explain how physical things affect our senses as they do. For a Berkeleyan idealist, on the other hand, the justification lies in the fact that, in speaking of material objects, we are doing no more than speaking about our sensations.

It is not certain that Frege would have allowed sense-perception as a justification for observational judgments, though, if not, it is obscure how there can be any non-deductive justifications for epistemology to investigate. On the hypothesis that he would have done so, our question must be whether the faculty by means of which we acknowledge synthetic *a priori* truths resembles reason or the perceptual faculties. In the last year of his life, Frege returned to the philosophy of mathematics which he had abandoned for eighteen years, ever since, in 1906, he had become convinced of the inadequacy of his attempted solution of Russell's paradox and thereby of the failure of his reduction of arithmetic to logic. He bravely undertook a new unification of mathematics on a geometrical basis, and wrote for publication an article on the sources of mathematical knowledge. In this he distinguishes three sources of knowledge: sense-perception; the logical source; and the geometrical and temporal source. Sense-perception is not needed for mathematical knowledge. The geometrical source is, of course, that from which the axioms of geometry derive, and only it or the temporal source, and neither the logical one nor sense-perception, can furnish us with the infinite.[30] The fragment 'Zahlen und Arithmetik', of the same date, identifies knowledge derived from the geometrical source as *a priori*;[31] and, as already observed, a final fragment identifies the geometrical source of knowledge with intuition, and, in doing so, retracts the declaration in *Grundgesetze* that 'arithmetic does not need to draw any ground of proof . . . from intuition'.[32]

In these very late writings, then, it is intuition on which our knowledge of synthetic *a priori* truth is based, and, specifically, spatial and temporal intuition. That does not, in itself, settle the question whether, in this last period, Frege regarded intuition as *justifying* such knowledge or simply as *consisting in* the ability to attain it: he asserts the existence of a geometrical source of knowledge, but says little to explain its nature, and it is therefore difficult to be sure in just what sense he is using the word 'intuition'. Friedrich Kaulbach has argued that his identification of intuition with the geometrical source of knowledge indicates that the latter is closely connected with the 'sensible *a priori* intuition of which Kant spoke and which he designated one of the "sources" from which flows the knowledge not only of geometry but also of arithmetic'.[33] Kant indeed says that 'time and space are two sources of knowledge [*Erkenntnisquellen*] from which different synthetic *a*

priori cognitions can be drawn'.[34] Kaulbach is very likely right that, in speaking of his three 'sources of knowledge', Frege was intending an allusion to Kant. He had made an express reference to Kant when, forty-two years before, he had used the same expression in a letter to Anton Marty.[35] This is not at all to say that, in these late writings, Frege was in complete agreement with Kant. Kant did not think, as Frege plainly implies, that all our synthetic *a priori* knowlege is derived from intuition. He did think intuition necessary for *self-evident* synthetic *a priori* truths: that is why mathematics has axioms, but philosophy does not, and why the categories require a transcendental *deduction*, while the axioms of geometry and the fundamental truths of arithmetic stand in no such need.[36] In Frege's late writings, on the other hand, not only does all synthetic *a priori* knowledge rest on the geometrical and temporal source, that is, on intuition, but there is no hint of the possibility of anything resembling a transcendental deduction.

4

In *Grundlagen*, equally, a transcendental deduction could find no place in Frege's account, since he says explicitly that all *a priori* propositions either can be deductively derived from, or themselves are, truths incapable of proof. It might be argued that, from Frege's standpoint, Kant's deductions of the categories do rest upon an unproved premiss, namely that experience is possible at all; but it would be special pleading indeed to try in such a manner to find room in Frege's theory for transcendental deductions of the Kantian type. The difference between the *Grundlagen* account and that given in Frege's writings of 1924 lies in the absence, from *Grundlagen*, of any suggestion that all synthetic *a priori* knowledge rests on intuition; it would be implausible to claim the one specific example cited by Frege of a fundamental non-logical law – the principle underlying induction – as so founded. What matters for present purposes, however, is not how far Frege disagreed with Kant about other synthetic *a priori* knowledge, but how much he agreed with him about geometry; and he does, in *Grundlagen*, repeatedly connect geometrical truths with intuition.

Frege made this connection in his earliest writings. His doctoral dissertation of 1873 begins with a Kantian enough statement that 'the whole of geometry rests ultimately on axioms which derive their validity from the nature of our intuitive abilities';[37] this raises the problem how we are able to speak of imaginary points of intersection of a circle with a straight line, or of points at infinity, which do not occur in the space of our intuition. In his Habilitationsschrift Frege wrote that

> there is a remarkable distinction between geometry and arithmetic as regards the basis of their fundamental principles. The elements of all geometrical constructions are intuitions, and geometry appeals to intuition as the source of its axioms. Since the subject-matter of arithmetic is not intuitable, its fundamental principles cannot spring from intuition.[38]

In the lecture 'Über formale Theorien der Arithmetik', Frege argued for the logical character of arithmetic, contrasting it with geometry; the negation of certain geometrical axioms would be logically possible, that is to say, would involve no contradiction. Arithmetic has no special domain of applicability, but extends to everything thinkable, whereas geometry applies only to what is spatial. If arithmetic were not reducible to logic, the ground of our knowledge of its correctness would

be in question: it could not be spatial intuition, for then arithmetic would be restricted to the geometrical; nor could it be physical observation, for it would then apply only to the physical.[39]

This contrast is drawn in much the same way in *Grundlagen*, §14. Arithmetical laws are neither empirical nor synthetic, being distinguished both from empirical propositions and from geometrical truths by their range of applicability. Empirical propositions hold good only of physical and psychological reality.[40] Geometrical truths govern the domain of what is spatially intuitable, both reality and the products of pictorial imagination. So long as they remain intuitable, the wildest flights of fancy are still subject to the axioms of geometry. Conceptual thought, however, can break loose from these axioms, as when it assumes a space of four dimensions or of positive curvature.[41] Such reflections, though far from useless, 'leave the ground of intuition behind': if we do make use of intuition in this connection, 'it is still the intuition of Euclidean space, the only one of which we have a picture'. For purposes of conceptual thought we can always postulate the opposite of one or another axiom of geometry without inconsistency: this proves both that those axioms are independent of one another and that they are synthetic. By contrast, we cannot deny any of the basic laws of arithmetic without falling into a confusion in which thought is no longer possible. The basis of arithmetic lies deeper, not only than that of the empirical sciences, but also than that of geometry: the domain of arithmetical truths is the widest of all, for it embraces not only what is real, like that of empirical science, nor even only what is intuitable, like that of geometry, but everything that is thinkable.

In *Grundlagen*, §90, Frege concedes that he has not so far conclusively demonstrated the analytic character of arithmetical propositions: that can be done only when proofs are completely formalized so as to preclude the occurrence of any gaps in them. In common mathematical practice, the mathematician is content so long as each transition in the proof is self-evident, without inquiring whether this self-evidence is logical or intuitive. Since the steps within the proof are longer than the simplest possible steps into which they could be broken down, two opposite mistakes are liable to be made. One is to take such a complex single step to be purely logical, when in fact some element of intuition has crept into it, the step representing a combination of simple inferences and of axioms of intuition. The other occurs when a complex inference involves no intuitive element, being applicable beyond the realm of what can be intuited, but we wrongly regard its self-evidence as intuitive, and accordingly rate the truth derived by means of it as synthetic. This can happen because, while its correctness is self-evident to us, we recognize that it does not conform to any of the standard forms of logical inference, and have failed to analyse it into its simple component steps. Relying only on informal proofs, we shall not succeed in distinguishing what is synthetic and rests on intuition from what is analytic, nor in compiling a complete list of axioms of intuition from which, together with the laws of logic, every mathematical theorem can be derived. Frege goes on, in §91, to report his formalization of logical inference in *Begriffsschrift*, by means of which it is possible to ensure that no unnoticed premiss creeps into any proof. He gives as an example of a theorem which might at first sight be taken to be synthetic, but which he had proved 'without borrowing any axiom from intuition', the proposition that the ancestral of a many–one relation is a simple ordering when restricted to the objects to which a given individual is ancestrally related. His conclusion is that 'the contents of sentences which extend our knowledge can be analytic judgments', something already asserted in §88.

5

If we are to elicit from these allusions to geometry a definite doctrine concerning it, we must understand Frege's use, up to 1885, of the word 'intuition'. His insistence that geometrical laws apply, not only to what we actually observe, but to all that we can imagine shows plainly that he was not then using the word in the restricted sense of the 1897 'Logik', in which it is, rather, '*Vorstellung*' that applies to the products of visual imagination. He certainly did not intend, however, to use the term in any vague, indeterminate sense. In *Grundlagen*, §12, he objects to Hankel's use of the expression 'pure intuition of magnitude', and comments that 'we are all too ready to invoke inner intuition whenever we cannot produce any other ground of knowledge. But we have no business, in doing so, to lose sight altogether of the sense of the word "intuition"'. He then proceeds to cite the explanation of the term given by Kant in his *Logik*, namely that 'an intuition is an individual idea [*Vorstellung*] . . . , a concept a general . . . or reflective idea'. Although Frege thus appeals to Kant as against Hankel, he was very dissatisfied with Kant's terminology. In the second footnote to §27, he objects to Kant's use, for both intuitions and concepts, of the word '*Vorstellung*', observing that he associated both an objective and a subjective meaning with the word, and that, in consequence, 'he gave his theory a very subjective, idealist colouring and made it difficult to hit on what his true opinion was'. Frege was not, of course, objecting merely to a defective terminology on Kant's part, but to an unclarity of thought. Unfortunately, Frege's own terminology in *Grundlagen*, though an improvement on Kant's, was still defective, because he had not yet won through to his distinction between *Sinn* and *Bedeutung*; and the footnote in which he comments on Kant's use of '*Vorstellung*' is the plainest example of this. Throughout *Grundlagen*, there is an oscillation in his use of the word 'concept' (*Begriff*), to be observed in his earlier writings also: in some passages it means what he would later have called the *Bedeutung* of a concept-word, in others its *Sinn*. In the footnote the same ambiguity affects the word 'object' (*Gegenstand*) also: we find him surprisingly saying that 'objective ideas can be divided into objects and concepts'. Frege explains that he himself will reserve the word 'idea' (*Vorstellung*) for its subjective sense: but it is evident that, by 'objective idea', he here means what he would later have called the sense (*Sinn*) of an expression, so that an objective idea of an object is what figures in his later writings as the sense of a proper name; here, however, it is not distinguished from the object itself, the *Bedeutung* of the name.

By 'intuition' Kant meant 'cognition of an object'; as stated in the passage of the *Logik* quoted by Frege, the distinction between intuitions and concepts is that the latter are general, the former individual. The same principle of distinction is maintained in the *Kritik*: 'an objective perception is a *cognition*. This is either an *intuition* or a *concept*. . . . The former relates directly to an object and is individual; the latter relates to it indirectly by means of a characteristic [*Merkmal*] which can be common to several things.'[42] Kant's distinction between intuitions and concepts thus corresponds to Frege's distinction, among ideas in the objective sense, between objects and concepts. It is for this reason that Frege observes, in *Grundlagen*, §12, that, in the sense of 'intuition' explained by Kant in the *Logik*, 'we might perhaps be able to call 100,000 an intuition; for it is certainly not a general concept'.

'An intuition in this sense, however', Frege goes on to say, 'cannot serve as the ground of our knowledge of arithmetical laws.' What is missing from the definition given in the *Logik* is any mention of a 'relation to sensibility, which, on the other

hand, is included in the notion of intuition in the "Transcendental Aesthetic"'. He quotes Kant as there saying, 'It is by means of sensibility that objects are *given* to us, and it alone furnishes us with intuitions',[43] and concludes that 'the sense of the word "intuition" is wider in the *Logik* than in the "Transcendental Aesthetic"'. It is only in the narrower of the two senses, that involving sensibility, that intuition 'can serve as the principle of our knowledge of synthetic *a priori* judgments'.

Since Frege himself believed intuition to be the basis of our knowledge of certain *a priori* truths, namely geometrical (though not, of course, arithmetical) ones, he evidently attached to the word 'intuition' the narrower of the two senses. This, however, creates a problem for Kant's classificatory scheme: how do numbers fit into it? As Frege remarks in *Grundlagen*, §104, natural numbers, at least very large ones like $1000^{1000^{1000}}$, are not intuitable: hence, as he says in §12, our awareness (or objective idea) of a number is neither a concept nor an intuition in the narrower sense;[44] and, as he says in §89, the number itself is not a concept, but an object, which, however, is not given to us either in sensation or in intuition. This shows that Kant was wrong in asserting that 'without sensibility no object would be given to us', and also in requiring of concepts that we should attach their objects to them in intuition,[45] unless, indeed, he used the word 'object' in a different sense from Frege. This last supposition would not resolve the difficulty: this would now take the form that there is no place for numbers in Kant's scheme.

Thus, for Frege, as for Kant, an intuition is of something particular or individual, and, as for Kant, it involves some relation to our sensory awareness; but, just because of this second feature, it cannot be equated with our awareness of objects of every kind. It is on the ground of the great generality of the notion of magnitude that Frege objects, in *Grundlagen*, §12, to Hankel's speaking of an 'intuition of magnitude':

> [I]f we consider all the different things that are called magnitudes – numbers, lengths, areas, volumes, angles, curvatures, masses, velocities, forces, intensities of illumination, electric currents, etc. – we can well understand how they can all be brought under the single *concept* of magnitude; but the term 'intuition of magnitude' . . . cannot be recognized as appropriate.

Furthermore, intuition involves not only particularity but also immediacy. As already noted, Kant held that a synthetic *a priori* truth can be self-evident to us only when it is founded upon an intuition; and Frege likewise requires a truth based on intuition to be immediately evident, though not conversely. In *Grundlagen*, §5, he remarks that Kant regarded numerical equations as unprovable and synthetic, though hesitating to call them axioms, since they are not general and there are infinitely many of them.[46] He goes on to observe that 'Kant wishes to call the intuition of fingers or of points in aid,[47] thus running the risk of making these propositions appear empirical, contrary to his own opinion; for the intuition of 37,863 fingers is certainly not a pure one'. The clinching refutation is that, if we had such an intuition, and others of 135,664 and of 173,527 fingers, the correctness of the equation '135,664 + 37,863 = 173,527' would have to be 'immediately evident, at least for fingers', which it is not.

The immediacy of intuition is reflected, not only in our immediate recognition of truths derived from it, but also in the unmediated application of the notions in terms of which we characterize it. That is why, in the section just quoted, Frege denies that we have any intuition of 135,664 fingers. The same point underlies his discussion of the notion of direction in §64 of *Grundlagen*. Frege remarks that 'many scholars give the definition: parallel straight lines are those which have the

same direction'. It is possible that he was here thinking, among others, of Hermann Lotze, who, in §131 of his *Metaphysik*, wrote that 'we call parallel two straight lines *a* and *b* which have the same direction in space'; if so, Frege is being slightly unfair, since Lotze goes on to give a criterion for having the same direction. In any case, Frege's comment on a definition of this kind is that 'the true state of affairs is thereby turned on its head'. In arguing this, he first states that 'everything geometrical must originally be intuitable', and then goes on to ask whether 'anyone has an intuition of the direction of a straight line'. His answer is that we have an intuition of a straight line, but do not distinguish in intuition the direction of the line from the line itself; he concludes that 'direction' is not a primitive geometrical notion, but that 'this concept is discovered only by means of a process of mental activity which takes its start from intuition'. The upshot is that the term 'direction' requires definition: in defining it, we may use the relational expression 'is parallel to' because 'we do have an idea [*Vorstellung*] of parallel lines'. Frege's statement in the late fragment 'Zahl' that points, lines, and planes are not, properly speaking, perceptible by the senses[48] represents a more cautious attitude than his blithe statement in *Grundlagen* that we can have an intuition of a straight line, but is based on the same guiding principle.

This feature of intuitions, as Frege conceived them, comes out most sharply in the philosophical discussion of the concept of magnitude which introduces *Rechnungsmethoden*, a work which foreshadows his uncompleted treatment of real numbers in *Grundgesetze*, Part III. Frege there uses the argument from generality he was to use again against Hankel in *Grundlagen*: 'so comprehensive and abstract a concept as that of magnitude cannot be an intuition.'[49] More interestingly, he also argues that even specific types of magnitude, such as length, area, and size of angle, are not derived from intuition: these concepts have been gradually disentangled from intuition, the intuitability formerly ascribed to them having been only apparent. 'Bounded straight lines and plane surfaces enclosed within curves are indeed intuitable; but the notion of magnitude as applied to them, something common to lengths and areas, eludes intuition.' The key notion, in grasping the concept of magnitudes of a given kind, such as length, is that of adding two such magnitudes. Angles provide a clear example: one cannot convey to a beginner a correct idea of an angle just by showing him a figure; one has to show him how to add angles, and then he knows what they are.

We may thus characterize that notion of intuitions with which Frege operated up to 1885 as follows. An intuition is a direct presentation, after some sensory mode, of a particular object or collocation of objects. Sense-perception is undoubtedly one species of it: but since it also occurs when we form mental pictures, it need not be the apprehension of any actual object or collocation of them. It can, however, be only of what can be immediately recognized: what is intuited cannot be characterized by appeal to any concept which it requires a further mental operation either to attain or to acknowledge as applying to the particular case. Such a notion of intuition does not appear to have assumed a major role in Frege's thought during this early period, since he was principally concerned with logic and arithmetic, in which, for him, intuition plays no part; his observations about geometry are motivated by the desire to contrast it with arithmetic. It appears, nevertheless, to have been the notion of intuition which he employed: one that embodies a particular conception of how we arrive at certain concepts and how we recognize certain truths. If that is right, the conjecture that he later became dissatisfied with it should occasion no surprise.

6

In order to determine whether Sluga correctly interprets the views held by Frege during the period that includes *Grundlagen*, we have to extract from that and his earlier writings a coherent theory of our knowledge of geometry. It was reasonable for him to argue from the universal applicability of arithmetical truths to the reducibility of arithmetical notions to purely logical ones; indeed, his criterion in *Grundlagen*, §3, for a truth's being logical in character is precisely that it be of unrestricted application, so that, in default of any alternative characterization of logical notions, no step is involved in such an argument. The converse argument, not expressly advanced by Frege, from the fact that geometry applies only to what is spatial to the impossibility of defining geometrical notions in purely logical terms would, in itself, be fallacious. This is because the expressibility of arithmetical truths in purely logical terms and their being analytic are two independent theses: an axiom of infinity, for example, might be formulable in logical terms, and yet not analytic in the epistemic sense given to that word by Frege. He appealed to the fact that arithmetical truths apply to everything thinkable in support of their analyticity, as well as in support of their logical expressibility: for if they rested on sense-perception, they would apply only to the actual world, and if on intuition, then only to what is intuitable. This is an epistemological argument, concerned with the grounds of our knowledge. Suppose that there were some synthetic truth, say A, which was nevertheless expressible in purely logical terms. Then, if we knew A at all, we could not know it in virtue of our logical faculties alone, but on the basis of observation or by appeal to intuition. Because we knew it in this way, we should not take it as holding good of everything thinkable; but, by hypothesis, we should be mistaken if we inferred from that that it could not be expressed in logical terms. It would therefore involve a fallacy to infer from the fact that geometrical truths relate only to what is spatial that they cannot be expressed in purely logical terms. It is nevertheless evident that they cannot. That does not imply, under Frege's definitions of 'analytic' and 'synthetic', that they are synthetic: a deep logical analysis might uncover interrelations between the different geometrical notions that could be embodied in a system of definitions by appeal to which the axioms of geometry could be shown to be analytic. For this reason, the converse of Frege's argument from the universal applicability of arithmetic to the analyticity of its truths could not be used to show geometry to be synthetic. There are two distinct reasons why a truth may not be universally applicable. It may involve notions whose range of application is restricted; or our knowledge of it may rest on other than purely logical grounds. These reasons are independent: there can be no valid inference from either to the other.

What does show the synthetic nature of geometrical propositions is the logical consistency of a denial of the axioms of geometry. Frege later became dubious about the proofs of independence of the Euclidean axioms;[50] but whether or not the possibility of *proving* the consistency of non-Euclidean geometries be admitted, to assert that the Euclidean axioms are synthetic is equivalent to asserting, as Frege did in *Grundlagen*, §14, that their negations are logically consistent. That they are synthetic is shown by the fact that we can consistently describe a world in which they fail; that we cannot *imagine* such a world shows that they are nevertheless *a priori*, since, if they rested upon empirical observation, they would apply only to the actual world, not to everything imaginable. That, at any rate, appears to be the argument that Frege is implicitly advancing. As he wrote in his doctoral dissertation, their validity derives from the nature of our intuitive abilities.

The particularity of intuitions seems to present an obstacle to their being the foundation of *general* laws such as underlie all synthetic *a priori* truths and are embodied in the axioms of geometry. Frege resolves this difficulty in *Grundlagen*, §13. One of the differences between arithmetic and geometry is that each number has an individual character of its own, whereas a geometrical point, line, or plane cannot, in itself, be distinguished from any other point, line, or plane:

> [I]t is only when several points, lines, or planes are simultaneously grasped in a single intuition that one distinguishes them. When in geometry general propositions are derived from intuition, it is evident from this that the points, lines, or planes that are intuited are not really particular ones and hence can serve as representatives for the whole of their kind.

What makes it appropriate to speak of intuition here is that the spatial configuration is presented to us as a particular one; but we apprehend in it only such features as are invariant under Euclidean transformations, and hence can base general laws upon it.

If our *a priori* knowledge of geometry rests upon intuitions, those intuitions must themselves be *a priori*, pure intuitions in the sense in which Kant explains that he calls all ideas *pure* in which 'nothing is met with that belongs to sensation'.[51] In *Grundlagen*, Frege uses the term 'pure intuition' only to deny that it applies in certain cases.[52] In §12, however, he remarks that Kant, having opted for the synthetic *a priori* character of arithmetical laws, had no alternative 'but to invoke a pure intuition as the ultimate ground of our knowledge'; it would seem to follow that Frege was under the same compulsion in respect of geometry. Kant, in the passage just cited, says that 'the pure form of sensible intuitions' is found in the mind *a priori*, and that 'this pure form of sensibility is itself called a *pure intuition*'. His justification for calling it an intuition is, in so far as it assumes a spatial form, that he does not regard space either as a property of material things or as a system of relations between them, but thinks that we conceive of it as a kind of receptable, an all-embracing object containing them.[53] There is, however, no suggestion anywhere in Frege's writings that he shared Kant's conception of a single pure intuition of space as a whole. The intuitions upon which, for Frege, geometrical laws depend appear to be of particular spatial configurations; they are pure only in the sense that they may be constructed by the imagination and that it is irrelevant whether they are ever encountered in observation.

Frege claims to be able to show that the informal reasoning we employ in proofs of arithmetical theorems can be validated by rigorous analysis. In informal reasoning about geometrical propositions, on the other hand, we take steps that are not always deductively valid: they may appear so only because we advert to figures drawn on paper or visualized in the imagination. Certain logical possibilities cannot be represented in such an actual or mental picture, and we therefore leave them unconsidered. An exact logical analysis, appealing to a formalization of deductive inference, would expose the points at which we were making a surreptitious appeal to synthetic laws, and would enable us to systematize, as a set of axioms, the laws so appealed to. Our acceptance of such fundamental laws rests on the impossibility of our imagining them to be violated, and hence on *a priori* intuitions, where 'intuition' is used as comprehending both spatial perception and spatial imagination. Such intuitions are *a priori* in that the constraints that govern them do not depend upon our observation of any particular facts; that is why they continue to apply to all that we can imagine, not just to the world as we find it to be.

7

To hold our knowledge of geometrical truths to be founded upon intuition no more entails adopting a transcendentally idealist view of space than does regarding them as synthetic *a priori*. Once more, all depends upon what explanation, if any, is given for intuition's yielding such knowledge. This is, indeed, the weak point in the account of geometry which we have seen to be implicit in the passing remarks Frege made about it up to 1885. The transcendental idealist explains the matter by treating the ground of our knowledge as also being the ground for the truth of the propositions known, that is, by identifying that whereby we know them with that in virtue of which they are true. On Kant's view, to have admitted a gap between these would deprive us of any reason to take synthetic *a priori* propositions, in particular the laws of geometry, to be true.

> Assume, then, that space and time are objective in themselves, and are conditions of the possibility of things in themselves, . . . Since the propositions of geometry are known synthetically *a priori*, . . . I ask: from where do you obtain such propositions, and on what does our understanding rest, in order to attain such absolutely necessary and universally valid truths? . . . It is therefore indubitably certain . . . that space and time . . . are merely subjective conditions of all our intuition.[54]

This is why, for Kant, geometrical laws hold only of the phenomenal world, of the world as it appears to us. Space is not objective, not a feature of things as they are in themselves; rather, we construct it from our *a priori* intuitions and impose it on the external world, which, in consequence, is only the world of appearances and not reality as it is in itself and independent of us.

Such an account would accord very badly with Frege's general outlook as expressed in the Preface to *Grundgesetze*, in which, as we have seen, he insists upon the gap between being true and being taken to be true. This was not a new attitude on his part: in the 'Logik' of the 1880s, he says, in just the same spirit, that the sense of the word 'true' precludes any reference to the knowing subject.[55] Is it not possible, nevertheless, that his attitude was less inflexible at the time of writing *Grundlagen*? As Kant says, if we assume that space is a feature of an objective reality independent of us, it is hard to see why 'the nature of our intuitive abilities' should afford us any sure guide to its constitution. It may have been the difficulty of answering this question that led Sluga to assume that Frege must have shared Kant's transcendental idealism. A realist must either refuse to offer any explanation, or assume an intrinsic harmony between independent reality and the form of our intuitions. It is conceivable that Frege would have denied that any justification could be given for fundamental non-logical laws any more than for the fundamental laws of logic; but such a position would undeniably be even less satisfactory in the former than in the latter case. The laws of logic are the laws of the laws of nature,[56] laws of truth with which we must comply if our judgments are to be true.[57] Even if, then, we can in certain cases give no answer to the question why we take a given law to be a law of logic, there is no general problem with what right we assume the laws of logic to be objectively true. Physical objects, as being actual (*wirklich*), act upon the senses;[58] it is therefore reasonable to take the sense-perceptions to which they give rise as indications of their existence and their properties. Moreover, as Frege observes in his 'Erkenntnisquellen' of 1924, sense-perception yields knowledge only when corrected by our knowledge of natural laws, which, he says, depends in part upon the logical and geometrical sources of knowledge and enables us to detect illusions.[59] To maintain, in the context of a

realist philosophy, that *a priori* intuitions yield genuine knowledge of objective reality is far more problematic. Synthetic *a priori* laws are not laws of truth itself; if the intuitions on which rests our acceptance of geometrical laws as true are *a priori*, it is not physical reality which has directly given rise to them: why, then, should we take them as revealing the truth about that reality?

According to Frege, we cannot imagine space as other than three-dimensional and Euclidean; *a fortiori*, we cannot perceive it otherwise. The truth, and even the meaning, of this contention are open to question. In one sense, our visual perception of space is not three-, but two-dimensional; it is reasonable to say that, in that sense in which we perceive it as three-dimensional, we have learned to do so, and hence, if our experience had been different, we could have learned to perceive it as four-dimensional.[60] Frege also held it to be logically possible for space not to conform to the laws of three-dimensional Euclidean geometry. We may therefore intelligibly ask how things would be for us if it did not. Frege never asked this question; any answer to it weakens his position, if that position is interpreted as realistic. One answer would be that we should fall into complete confusion, continually misinterpreting our sensations or unable to interpret them at all. Another would be that, in such a case, we should both imagine and perceive space as conforming to whatever was the true geometry. Yet a third answer would be that we should continue to perceive space as three-dimensional and Euclidean, but that we should be able to discover its true character and to arrive indirectly at correct judgments about the spatial disposition of objects. Given Frege's assumption that three-dimensional Euclidean geometry correctly describes actual physical space, any one of these three answers would furnish us with *a posteriori* grounds for accepting that assumption. A fourth possible answer is that, if the geometry of physical space were, say, elliptic, or if it had four dimensions, we could never become aware of the fact. This answer would deprive us of any ground whatever for taking physical space in fact to be as our intuitions represent it. It is to this dilemma that the doctrine that geometry is synthetic *a priori* and rests on intuition leads, when understood against the background of a realist view of the physical universe. Frege's logicist theory of arithmetic is not caught in this fork. One cannot argue against it that, if the laws of arithmetic did not hold, then either we should be aware of their failure, in which case arithmetic is an empirical science, or it would make no difference to us, in which case we have no reason for believing those laws to hold: for, on Frege's account, the failure of arithmetical laws is a logical impossibility. The thesis that geometry is synthetic *a priori is* impaled upon the fork: it is unclear that Frege was ever conscious of the fact.

This, then, is a good prima facie reason for regarding the view of geometry suggested by Frege's scattered remarks about it in *Grundlagen* and elsewhere as untenable save in the context of transcendental idealism. It is not, however, a good reason for interpreting Frege as having subscribed to that or any other form of idealism. This is apparent from *Grundlagen*, §26, of which we have so far taken no account, and which includes Frege's most sustained discussion of space. He begins by remarking that 'according to Kant, space belongs to appearance'.[61] 'For other rational beings', he continues, 'it might take some form quite different from that in which we know it.' This is not a Kantian thesis. Kant says that it is possible that all thinking beings must necessarily have a mode of spatial and temporal intuition like the human mode, but that we cannot decide the question;[62] but Frege is asserting the thesis on his own account. He goes on to say that 'we cannot even know whether [space] appears the same to one man as to another; for we cannot

lay one man's intuition of space beside another's in order to compare them'. 'Nevertheless', he declares, 'there is something objective in space all the same.' 'What is objective in it', he says, 'is what is subject to laws, what can be conceived, what can be judged,[63] what can be expressed in words. What is purely intuitable[64] is not communicable.' He justifies this by saying that 'everyone recognizes the same geometrical axioms, even if only by his behaviour, and must do so if he is to find his way about the world'; and he proceeds to illustrate it by an example. He supposes two rational beings for whom only projective properties and relations are intuitable; and he supposes further that what one intuits as a point appears to the other as a plane and conversely, so that what for one is the line joining two points is for the other the intersection of two planes, and so on. He comments that 'they could understand one another very well'; because of the principle of duality in projective geometry, 'they would never become aware of the difference in their intuitions'. In particular,

> they would be in complete agreement concerning all geometrical theorems; they would merely translate the words differently into their intuitions. With the word 'point', for example, one would connect this intuition, the other would connect that one. We can therefore still say that this word means something objective for them; it is just that we must not understand by this meaning anything peculiar to their intuitions.

Frege goes on to compare this case with that of colour.

> The word 'white' ordinarily makes us think of a certain sensation, which is of course wholly subjective; but even in ordinary linguistic usage, it seems to me, an objective sense frequently predominates. When we call snow white, we wish to express an objective state which we recognize, in ordinary daylight, by a certain sensation. When the light is coloured, we allow for that in the judgment that we make: we say, for instance, 'It *appears* red at present, but it *is* white'. Even a colour-blind man can speak of red and green, although he does not distinguish these colours in sensation; he recognizes the distinction by the fact that others draw it, or perhaps by means of a physical experiment. Often, therefore, the colour-word does not signify our subjective sensation, of which we cannot know that it agrees with that of someone else, . . . but an objective state.

The whole discussion is summed up by a restatement of what Frege means by 'objective':

> [B]y objectivity I thus understand independence from our sensation, intuition and imagination [*Vorstellen*], and from the delineation of inner pictures from memories of earlier sensations, but not independence from reason; for to answer the question what things are independently from reason would be to judge without judging, to wash the fur without wetting it.

According to Frege, then, we associate certain sensations or intuitions with certain words, such as 'white' or 'point'; but we cannot rely upon such associations to convey our sensations to someone else, since he may connect a different sensation or intuition with the same word, and we can never know that this is not so, sensations and intuitions being subjective and incommunicable. Hence, in order to communicate successfully, we must accord to our words an objective sense, one that is independent of our own sensations and intuitions. Successful communication occurs when there is agreement on how to judge the truth of what is expressed by the words; a word such as 'point' or 'white' thus bears an objective sense when it is used in sentences in such a way as to convey by means of them an objective state of affairs, about which all agree how to determine whether or not it obtains. Space

is objective in so far as we can state its properties in words bearing an objective meaning; and the axioms of geometry state just such properties. Everyone must acknowledge their truth, even if only implicitly, in his actions; in doing so, he acknowledges what holds objectively and can be wholly grasped by conceptual thought independently of intuition or imagination.

Many objections can be brought against this argument of Frege's: that it rests on an untenable view of the privacy of inner sensations; or that, to move about the world successfully, we need assume at most that space approximates to being Euclidean in small regions, just as the surface of the Earth approximates to a plane over small areas. But §26 is no aberration on Frege's part. Given his unwavering conviction of the ineradicable subjectivity of ideas, where 'idea' is understood in the generic sense, as covering sensations, intuitions, and mental images, he could say no other; the alternative would be to allow that a geometrical axiom might be true for one person and false for another. It might seem surprising, in view of Frege's various statements that it is on intuition that our knowledge of geometrical axioms rests, that he should say in §26 that their contents relate only to those features of space which are independent of sensation and intuition. The surprise is merely superficial. We cannot imagine what it would be like for the axioms of geometry to be false, but we can conceive of their falsity, that is, we can think their negations: it follows that their senses are capable of being wholly grasped by conceptual thought in a manner that involves no allusion to our intuitions. It is on the basis of *a priori* intuitions of space that we accept those axioms as true; but the features of those intuitions which the axioms capture are ones which, as being expressible in words, are common to all and could, therefore, be grasped even by a subject whose intuitions differed from ours.

None of this, admittedly, extricates Frege's account from the fork, 'either empirically based or groundless', discussed above; in itself, it is even compatible with a transcendentally idealist account of what makes geometrical propositions true. The passage was not, however, written without Kant in mind: it opens with a reference to him, and Frege was as concerned here as throughout *Grundlagen* to relate his views to Kant's. Kant quite rightly said that his explanation of why our *a priori* intuitions of space and time are sources of knowledge entails that 'space and time . . . are merely subjective conditions of all our intuition' and that 'as phenomena [*Erscheinungen*], they cannot exist in themselves, but only in us'.[65] Hence, when Frege repudiates Kant's view that space belongs to appearance (*Erscheinung*), and opposes to it the thesis that it is in certain respects, including those expressed in the axioms of geometry, objective, where objectivity involves independence from intuition, he can only be meaning to reject Kant's account of what makes the laws of geometry true. From the 'Logik' of the 1880s onwards, Frege always understood 'objective' in the same way as Kant, namely as meaning what is altogether independent of us: thus in the 1897 'Logik' he characterizes it as 'that which is independent of our mental life', and says later that 'thoughts do not belong like ideas to the individual mind (are not subjective), but are independent of thinking, and stand (objectively) over against everyone in the same way'.[66] In speaking of space as objective, therefore, Frege must be taken as asserting that it possesses those of its properties which are expressed in words and stated by geometrical laws independently of our mode of apprehending it. It is not, for him, our recognizing those laws as true that makes them true; we do not construct physical space, let alone the physical universe, any more than we construct the numbers, but, in both cases, give a name to what is there independently of us. In

speaking about space, as described by geometry, we are not, for Frege, speaking of our *intuitions* or sensations. If this leaves it obscure why we should treat our intuitions as a ground of knowledge of geometry, that is a lacuna that Frege might have filled had he ever written positively about geometry. It is also perhaps a reason why, from 1885 to 1924, he ceased to assert that our knowledge of geometry is founded upon intuition: but it is not a ground for construing him to have meant the opposite of what he said.

In the unfinished 'Logik' of the 1880s, probably written close to the time when *Grundlagen* was composed, Frege explained why he called the Sun 'objective' as follows:

> Is not the Sun for some people a beneficent or malignant deity, for others a shining disk hurled into the heavens from the East and rolling down again towards the West, for yet others an immense spherical white-hot body enveloped by a cloud of incandescent gases? No. To one person it may *appear* one thing, to another, another: it *is* what it is.[67]

One who subscribed to a transcendentally subjective view of physical objects could find a way of assenting to that; for he could explain that what the Sun *is*, as here contrasted with how it *appears*, is still only a matter of appearance, of how things are in the phenomenal world. The contrast drawn between its objective and its apparent character, he could say, is really only that between intersubjective judgments on which all can come to agree and personal ones which are to be abandoned.[68] In this way, such a one could assent to the passage while still maintaining that physical objects are mere appearances: it would be psychologically impossible for him to have written it.

NOTES

1 H. Sluga, 'Frege's Alleged Realism', *Inquiry*, Vol. 20 (1977), pp. 227–42; see p. 237.
2 Ibid., p. 236.
3 Hans Sluga, *Gottlob Frege*, in the series *The Arguments of the Philosophers*, ed. by Ted Honderich, Routledge & Kegan Paul, London/Boston/Henley 1980, p. 45.
4 An exception is his letter to Marty of 1882, in which he uses the terms 'analytic' and 'synthetic' (*Wissenschaftlicher Briefwechsel* [*Briefwechsel*], ed. by G. Gabriel *et al.*, Felix Meiner, Hamburg 1976, p. 163, *Philosophical and Mathematical Correspondence*, Blackwell, Oxford 1980, pp. 99–100); he was here expounding *Grundlagen*, which he was in the course of writing. Another exception is the reference to *a priori* knowledge in the 'Zahlen und Arithmetik' of 1924–5 (*Nachgelassene Schriften*, Felix Meiner, Hamburg 1969, p. 297, *Posthumous Writings*, Blackwell, Oxford 1980, p. 277).
5 Save for a brief allusion at the very end of his 'Erkenntnisquellen' of 1924–5 (*Nachgelassene Schriften*, p. 294, *Posthumous Writings*, p. 274).
6 *Die Grundlagen der Arithmetik (Grundlagen)*, Breslau 1884, §87.
7 *Grundgesetze der Arithmetik (Grundgesetze)*, Vol. I, Jena 1893, p. 1.
8 *Grundlagen*, §89.
9 In P. E. B. Jourdain, 'The Development of the Theories of Mathematical Logic and the Principles of Mathematics: Gottlob Frege', *The Quarterly Journal of Pure and Applied Mathematics*, Vol. XLIII (1912), p. 241. The original German version of Frege's notes is contained in a letter to Jourdain in *Briefwechsel*, op. cit., pp. 114–24.
10 *Grundgesetze*, Vol. I, p. 1.
11 *Nachgelassene Schriften*, p. 142.
12 *Posthumous Writings*, p. 131.
13 *Briefwechsel*, p. 70, *Philosophical and Mathematical Correspondence*, p. 43.
14 'Über die Grundlagen der Geometrie', *Jahresbericht der deutschen Mathematiker-Vereinigung*, Vol. XII (1903), p. 319.

15 'Neuer Versuch der Grundlegung der Arithmetik', *Nachgelassene Schriften*, p. 298, *Posthumous Writings*, p. 278.
16 *Nachgelassene Schriften*, p. 285, *Posthumous Writings*, pp. 265–6.
17 *Grundlagen*, §§26, 61, 85, 109; *Grundgesetze*, Vol. I, p. xviii.
18 *Grundlagen*, §96.
19 'Logik', 1880s, *Nachgelassene Schriften*, p. 7, *Posthumous Writings*, p. 7.
20 'Logik', 1897, *Nachgelassene Schriften*, p. 144, *Posthumous Writings*, p. 133.
21 *Grundlagen*, §3. Strictly speaking, I have in the text stated the alternatives too crudely. Restricting our attention to proofs involving no inference that discharges a hypothesis, let us call such a proof 'irredundant' if, when set out in tree form, no proposition appears more than once on any branch; and let us call a sequence of irredundant proofs an 'extension chain' if each term is a (proper) extension upwards of its predecessor. Then the relevant possibilities are: (a) that there are infinite extension chains; and (b) that every extension chain is finite. On hypothesis (b), it might be true that every individual proposition we can know to be true was capable of proof; but we could still comply with the yet weaker admonition, 'Prove *as much* as can be proved', which we could not do on hypothesis (a). From remarks in Frege's later writings, it is apparent that it is hypothesis (b) that he is really concerned to maintain, rather than the stronger thesis that there are specific propositions which cannot be proved but can be known; see in particular the discussion of axioms in 'Logik in der Mathematik', *Nachgelassene Schriften*, pp. 221–2, *Posthumous Writings*, pp. 205–6.
22 Austin's translation, 'if we recognize the existence of general truths at all', involves a mistake. The point is not that we acknowledge that there are some true general propositions, but that we acknowledge certain general propositions as true.
23 'Logik', 1880s, *Nachgelassene Schriften*, p. 3, *Posthumous Writings*, p. 3.
24 *Grundgesetze*, Vol. I, p. xvii. Furth makes one of his rare slips in translation at this point, rendering the second half of the second sentence as 'logic can give no answer'. This might leave it open that some other science or branch of philosophy could give the answer: but Frege says explicitly that the question must remain unanswered (*schuldig*).
25 Ibid., p. xvi.
26 Ibid., p. xvii.
27 Ibid., p. xv.
28 *Grundgesetze*, Vol. II, Jena 1903, §§74, 147.
29 'No . . . description of the inner processes which precede the delivery of a judgment . . . can ever be adduced in proof.' *Grundlagen*, §26.
30 'Erkenntnisquellen der Mathematik', *Nachgelassene Schriften*, pp. 286–94, *Posthumous Writings*, pp. 267–74.
31 *Nachgelassene Schriften*, pp. 296–7, *Posthumous Writings*, pp. 276–7.
32 See Notes 10 and 15, above.
33 *Nachgelassene Schriften*, p. xxxi.
34 *Kritik der reinen Vernunft* (*Kritik*), B 55.
35 Mentioning that he has nearly completed a book, evidently *Grundlagen*, he asserts what he also says there (§§88–89), that Kant placed too low a value on analytic judgments, that arithmetical truths are analytic, not synthetic as Kant thought, but that Kant deserves great credit for having recognized geometrical propositions as synthetic. 'The two cases are quite different. The domain of geometry is that of the spatially intuitable; arithmetic knows no such restriction. . . . The domain of what is countable is . . . as broad as that of conceptual thought, and a source of knowledge [*Erkenntnisquelle*] of more restricted extension, such as spatial intuition or sensory perception, would not suffice to guarantee the general validity of arithmetical propositions' (*Briefwechsel*, pp. 163–4, *Philosophical and Mathematical Correspondence*, p. 100).
36 *Kritik*, B 760–1.
37 *Über eine geometrische Darstellung*, 1873, p. 3.
38 *Rechnungsmethoden*, 1874, p. 1.
39 'Über formale Theorien der Arithmetik', *Sitzungsberichte der Jenaischen Gesellschaft für Medizin und Naturwissenschaft für das Jahr 1885*, Jena 1885, pp. 94–95.

40 *Wirklichkeit*: for once this word is better translated 'reality' than 'actuality', since Frege is concerned, not with the difference between that which is causally efficacious and that which is not, but with that between what exists and what is only imagined.
41 By the axioms of geometry Frege always understands those of three-dimensional Euclidean geometry: see the fragment 'Über Euklidische Geometrie', *Nachgelassene Schriften*, pp. 182–4, *Posthumous Writings*, pp. 167–9.
42 *Kritik*, B 376–7. Here a perception (*Perception*) is an idea of that particular kind which involves awareness (*Vorstellung mit Bewusstsein*). Such a perception may be either a sensation (*Empfindung*) or a cognition (*Erkenntnis*), the latter being subdivided into intuitions and concepts.
43 Ibid., B 33.
44 'I cannot allow an intuition of 100,000.'
45 Both remarks are cited from ibid., B 75, where Kant also says that 'our nature is so constituted that intuition can never be anything but sensible, i.e. it contains the only way in which we are affected by objects'.
46 The reference is to ibid., B 205.
47 An allusion to ibid., B 15.
48 See Note 16, above.
49 *Rechnungsmethoden*, p. 1.
50 Thus in a letter to Liebmann of 1900 he wrote, 'I have reason to believe that the mutual independence of the axioms of *Euclidean* geometry cannot be proved' (*Briefwechsel*, p. 148, *Philosophical and Mathematical Correspondence*, p. 91); and in his notes to Jourdain's article he says, 'The indemonstrability of the axiom of parallels cannot be proved'. See Jourdain, op. cit., p. 240.
51 *Kritik*, B 34.
52 *Grundlagen*, §§5 and 12 (in the latter against Hankel's 'pure intuition of magnitude').
53 *Kritik*, B 38–39, 42. How far this committed him to a full acceptance of absolute Newtonian space is open to argument. Frege maintained that Newton's absolute space was not wholly transcendent, but was connected with experience via the law of inertia; on the other hand, 'in Newton's assumption of a single absolute space more is contained than is necessary for the explanation of the phenomena' ('Über das Trägheitsgesetz', *Zeitschrift für Philosophie und philosophische Kritik*, Vol. MXVIII [1891], p. 149). In the context, this reads as an admission that Newton's hypothesis is partly, though not wholly, superfluous: but anyone determined to interpret Frege as a Kantian could read it to mean that it was *a priori*.
54 *Kritik*, B 64, 66.
55 *Nachgelassene Schriften*, p. 5, *Posthumous Writings*, p. 5.
56 *Grundlagen*, §87.
57 'Logik', 1897, *Nachgelassene Schriften*, p. 157, *Posthumous Writings*, p. 145.
58 *Grundgesetze*, Vol. I, p. xviii, *Grundlagen*, §85; cf. 'Der Gedanke. Eine logische Untersuchung', *Beiträge zur Philosophie des deutschen Idealismus*, Vol. I (1918), p. 76.
59 *Nachgelassene Schriften*, p. 287, *Posthumous Writings*, p. 268.
60 For a sketch of how this might be possible, see the excellent, and neglected, article by Honor Brotman, 'Could Space be Four-Dimensional?', *Mind*, Vol. LXI (1952), pp. 317–27.
61 See, e.g., *Kritik*, B 59–60.
62 Ibid., B 72.
63 '*Das Gesetzmässige, Begriffliche, Beurtheilbare.*'
64 '*Das rein Anschauliche.*'
65 *Kritik*, B 66, 59.
66 *Nachgelassene Schriften*, pp. 149, 160, *Posthumous Writings*, pp. 137, 148.
67 *Nachgelassene Schriften*, p. 7, *Posthumous Writings*, p. 7.
68 Thus Kant: 'We indeed ordinarily distinguish, among phenomena, that which belongs essentially to the intuition of them, and holds good for every human mind, from that which attaches only accidentally, in that . . . it holds only for a particular state . . . of this or that mind' (*Kritik*, B 62). In fact, however, 'we have to do with nothing but phenomena; . . . the transcendental object remains unknown to us' (B 63).

NOTE ADDED IN PROOF

To the citations in Section 1 should be added two more from an earlier letter of Frege to Hilbert, dated December 1899. The first (*Briefwechsel*, p. 61, *Philosophical and Mathematical Correspondence*, p. 35) is simply an allusion to Hilbert's statement, quoted in 'Über die Grundlagen der Geometrie' (1903), p. 321, that each of his five groups of axioms 'expresses certain corresponding basic facts of our intuition' (D. Hilbert, *Grundlagen der Geometrie*, 7th ed., Leipzig/Berlin 1930, §1, p. 2). Frege asks how the word 'point' should be understood, as Hilbert uses it, and comments, 'One first thinks of points in the sense of Euclidean geometry, and this is confirmed by the statement that the axioms express basic facts of our intuition'. In the second instance, however, Frege is speaking on his own account. He says, 'I call axioms propositions that are true, but which are not proved, because the knowledge of them flows from a source of knowledge [*Erkenntnisquelle*] which is quite different from the logical one, and which one may call spatial intuition' (*Briefwechsel*, p. 63, *Philosophical and Mathematical Correspondence*, p. 37). This last citation, which I had previously overlooked, somewhat weakens my conjecture that, in his mature period, Frege had become dissatisfied with the notion of intuition: it remains that this letter was written less than three years after the 1897 'Logik', and that, in the published articles, Frege carefully avoids any such remark.